

Cluster monomials in $\mathbb{C}[GL_n/N]$, a simplicial fan in the cone of semi-standard Young tableaux, and the Lusztig basis

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Abstract

We study the cluster monomials and cluster complex in $\mathbb{C}[GL_n/N]$. For we consider the *tableau basis* in $\mathbb{C}[GL_n/N]$. Namely, an element Δ_T of the tableau basis labeled by a semi-standard Young tableau T is the product of the flag minors corresponding to columns of T . Our main results state: (i) cluster monomials in $\mathbb{C}[GL_n/N]$ can be labeled by semistandard Young tableaux such that any cluster monomial has the form $\Delta_T +$ lexicographically smaller terms; (ii) such labeling distinguish the cluster monomials; (iii) for any seed of the cluster algebra on $\mathbb{C}[GL_n/N]$, we define a cone in $\mathbf{D}(n)$ ($\mathbf{D}(n)$ is the cone of semi-standard Young tableaux, $\mathbf{D}(n)$ is linear isomorphic to the Gelfand-Tsetlin cone) generated by tableaux which label the cluster variables of the seed, and these cones form a simplicial fan in $\mathbf{D}(n)$.

1 Introduction

One of the main motivation of S.Fomin and A.Zelevinsky for introducing cluster algebras [4] was the desire to provide a combinatorial framework to understand the structure of 'dual canonical bases' in (homogeneous) coordinate rings of various algebraic varieties related to semisimple groups. Several such varieties (Grassmann varieties and double Bruhat cells) carry a cluster algebra structure and certain special functions on that spaces (such as Plücker coordinates, generalized minors) correspond to distinguished elements called *cluster variables*.

For the general linear group $G = GL_n(\mathbb{C})$, and the subgroup $N \subset G$ of unipotent upper-triangular matrices, the coordinate ring of the *base affine space* for $GL_n(\mathbb{C})$, $\mathbb{C}[GL_n/N]$, is the set of regular functions on $GL_n(\mathbb{C})$ which are invariant under the action of N by the right multiplication. According to classical invariant theory, $\mathbb{C}[GL_n/N]$ is generated by the *flag minors*. (A flag minor Δ_I of a matrix $x = (x_{ij}) \in GL_n(\mathbb{C})$ is a minor occupying in rows in I and the first $|I|$ columns.) The ring $\mathbb{C}[GL_n/N]$ is one of prototypical examples of a cluster algebra ([10]). For the cases $n = 3, 4, 5$, the cluster monomials form a basis in $\mathbb{C}[GL_n/N]$ as in the vector space, but for $n \geq 6$ the cluster monomials fail to be a basis.

We study the cluster monomials and cluster complex in $\mathbb{C}[GL_n/N]$. For we consider the *tableau basis* in $\mathbb{C}[GL_n/N]$ (see, for example [9, 11]). The elements of the tableau basis are labeled by the semistandard Young tableaux. Namely, an element Δ_T of the tableau basis labeled a semistandard Young tableau T is the product of the flag minors corresponding to columns of T (for details see Section 3). Because of the Désarménien-Kung-Rota algorithm ([9]), the set of such monomials in flag minors is a basis in $\mathbb{C}[GL_n/N]$.

Consider the lexicographical order \succ on the variables x_{ij} , $i, j \in [n]$, (i indicates the row number and j the column number) $x_{ij} \succ x_{kl}$ if $i < k$ or $i = k$ and $j < l$. This order defines

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a total order on the monomials of $\mathbb{C}[x_{ij}]$. In particular, we get a total order on the Young tableaux through the total ordering of Δ_T 's.

The semi-standard Young tableaux filled in the alphabet $[n]$ can be viewed as integer point of the cone of **D**-tight arrays $\mathbf{D}(n) \subset \mathbb{R}^{n(n+1)/2}$ (see [6]). The cone $\mathbf{D}(n)$ is linear isomorphic to well-known cone of the Gelfand-Tsetlin patterns. For us it is convenient to work with the cone of semi-standard Young tableaux.

Our main results (Theorem M1, M2 and M3) state: (i) cluster monomials in $\mathbb{C}[GL_n/N]$ can be labeled by semistandard Young tableaux such that any cluster monomial has the form $\Delta_T +$ lexicographically smaller terms; (ii) such labeling distinguish the cluster monomials; (iii) for any seed of the cluster algebra on $\mathbb{C}[GL_n/N]$, we define a cone in $\mathbf{D}(n)$ generated by tableaux which label the cluster variables of the seed; these cones form a simplicial fan in $\mathbf{D}(n)$.

In the case the cluster algebra $\mathbb{C}[GL_n/N]$ is of finite type, that is the case for $n = 3, 4, 5$, the union of the cones of the fan coincides with the cone $\mathbf{D}(n)$, and thus the cluster monomials form a basis in $\mathbb{C}[GL_n/N]$.

For $n \geq 6$, the cluster algebra of $\mathbb{C}[GL_n/N]$ is of infinite type, and the union of the cones is only a part of the cone $\mathbf{D}(n)$.

We conjecture that the cluster monomials correspond to the *real* elements of the Lusztig basis. Namely, the specialization of the dual canonical basis at $q = 1$ is the *Lusztig basis*. Recall, that the dual canonical basis is a basis of the quantum deformation of $\mathbb{C}[GL_n/N]$. This is a distinguished basis which nicely behaves with respect to implementation of $\mathbb{C}[GL_n/N]$ as a representation of GL_n (namely, $\mathbb{C}[GL_n/N]$ is the direct sum of the irreducible representations of GL_n each taken with multiplicity one).

The Lusztig basis is a basis in $\mathbb{C}[GL_n/N]$ labeled by the integer point of the cone $\mathbf{D}(n)$. An element of Lusztig basis labeled by an integer point of $\mathbf{D}(n)$ has the form $\Delta_T +$ lexicographically smaller terms, where T is a semi-standard Young tableau corresponding to the point in $\mathbf{D}(n)$.

The Lusztig basis and the tableau basis are different. For example, for GL_3 , the Lusztig basis (discovered in 1985 by I.Gelfand and A.Zelevinsky ([12])) is the collection of monomials in $\Delta_1, \Delta_2, \Delta_3, \Delta_{12}, \Delta_{23}, \Delta_{13}$, and Δ_{123} which do not contain the product $\Delta_2\Delta_{13}$, while the tableaux basis is the collection of monomials in $\Delta_1, \Delta_2, \Delta_3, \Delta_{12}, \Delta_{23}, \Delta_{13}$, and Δ_{123} which do not contain the monomial $\Delta_1\Delta_{23}$.

An element of the Lusztig basis is *real* if its square belongs to the Lusztig basis. We call the semi-standard Young tableau *real* if the corresponding element of the Lusztig basis is real. Our conjecture is that cluster monomials are labeled by real semi-standard Young tableaux.

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2 Cluster skew-symmetric algebras

Since we are interested in the cluster algebra structure on $\mathbb{C}[GL_n/N]$, we remind necessary definitions for so-called skew-symmetric cluster algebras.

Let $G = (V(G), E(G))$ be a *quiver* (a directed multigraph) in which the vertex set $V(G)$ is partitioned into two subsets: a set V_1 of *frozen* vertices, and a set V_2 of *mutable* vertices. The (integer) edge multiplicity function is regarded as being skew-symmetric: if vertices u, v

are connected by α edges going from u to v (which are members of $E(G)$), we simultaneously think of these vertices as being connected by $-\alpha$ edges going from v to u . To each vertex v of G one associates a *cluster variable* x_v so that $\{x_v : v \in V(G)\}$ is a transcendence basis of a field of rational functions \mathcal{F} . Such a pair consisting of a quiver and a transcendence basis indexed by its vertices is said to be a *cluster seed*. Monomials in $x_v, v \in V(G)$ are called *cluster monomials*. The cluster seed can be mutated at any mutable vertex to produce a new cluster variable and a new seed. Applying mutations in all possible situations produce the set of cluster variables and these variables form a skew-symmetric cluster algebra [4].

The quiver and variables are modified by applying the following operations called cluster mutations. A *cluster mutation* μ_v applied at a mutable vertex $v \in V_2$ changes one variable, namely, x_v , and modifies the quiver G by the following rules. For a vertex v , denote $In(v) := \{v' \in V(G) : (v', v) \in E(G)\}$ and $Out(v) := \{v'' \in V(G) : (v, v'') \in E(G)\}$.

The quiver $\mu_v(G)$ has the same vertex set as G , $V(\mu_v(G)) = V(G)$, partitioned into frozen and mutable vertices in the same way as before. The edges $E(\mu_v(G))$ are obtained from edges $E(G)$ by the following rule:

- (i) the edges in $E(\mu_v(G))$ incident to the vertex v are exactly the edges in $E(G)$ incident to v but taken with the reverse direction;
- (ii) for each pair $v' \in In(v)$ and $v'' \in Out(v)$, form the edge (v', v'') in $E(\mu_v(G))$ whose multiplicity is defined to be $\gamma - \alpha \cdot \beta$, where $\alpha = w(v', v) \geq 1$ is multiplicity of the edge (v', v) in $E(G)$, $\beta = w(v, v'') \geq 1$ is that for (v, v'') , and $\gamma \in \mathbb{Z}$ is that for (v', v'') ;
- (iii) the other edges of $\mu_v(G)$ are those of G that neither are incident to v nor connect pairs v', v'' as in (ii).

For $u \neq v$, we put $\mu_v(x_u) := x_u$ and define $\mu_v(x_v) = x_v^{new}$ by the following rule:

$$x_v^{new} \cdot x_v = \prod_{v' \in In(v)} x_{v'}^{w(v', v)} + \prod_{v'' \in Out(v)} x_{v''}^{w(v, v'')}.$$

This gives the new seed: the quiver $\mu_v(G)$ and variables $\mu_v(x_u), u \in V(\mu_v(G)) = V(G)$.

Obviously, there holds $\mu_v^2 = id$.

The variables $x_v, v \in V_1$ do not change and are called *coefficients*. Let \mathcal{X} denotes the set of cluster variables. Then the cluster algebra is $k[x_v | v \in V_1]$ -subalgebra of \mathcal{F} generated by \mathcal{X} .

We need the following important results on cluster algebras. Namely, the Laurent phenomenon ([4]) and recently proven Positivity conjecture for skew-symmetric cluster algebras [16].

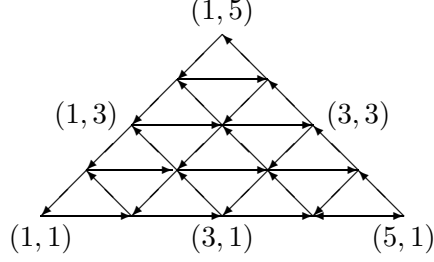
Theorem CL. ([4]) For any initial seed $(G; x_v, v \in V(G))$, any cluster variable $x \in \mathcal{X}$ is a Laurent polynomial in variables $x_v, v \in V$.

Theorem CP. ([16]) For a skew-symmetric cluster algebra, the coefficients in such Laurent polynomials are positive.

3 Cluster monomials in $\mathbb{C}[GL_n/N]$ and a fan in the cone of semi-standard Young tableaux

For given n , let us consider cluster algebras $\mathcal{A}(n)$ with the initial seed specified by the quiver $Q(n)$ being a triangular grid of size n with cyclically oriented triangles (we depicted $Q(5)$ below).

We are interested in such a cluster algebra, because the quiver $Q(n)$ corresponds to a pseudoline arrangement for the reduced decomposition of the longest permutation $w_0 = s_1 s_2 \dots s_{n-1} s_1 \dots s_{n-2} \dots s_1 s_2 s_1$ (see [10] and [2]).



For us it is convenient to label the vertices of $Q(n)$ by (i, j) , $i + j \leq n + 1$, $i, j \geq 1$, such that the vertex labeled by $(1, 1)$ is at the left corner of the triangular grid, and (i, j) labels the end point of any path in Q which has $i - 1$ horizontal edges directed to East and $j - 1$ slope edges directed North-East (note, that the north-east direction is the opposite to the direction of the corresponding edges in $Q(n)$). (We depicted the vertices $(1, 1)$, $(3, 1)$, $(1, 3)$, $(1, 5)$, $(5, 1)$ and $(3, 3)$ in the above picture.) The vertices at the left and right sides of the triangle are frozen, that is the set V_1 of the frozen vertices consists of vertices labeled by $(i, 1)$, $i = 1, \dots, n$, and (i, j) with $i + j = n + 1$, and the set V_2 of mutable vertices is the vertices labeled by $\{(i, j)$, either $i \neq 1$ or $i + j < n + 1\}$.

The cluster structure on $\mathbb{C}[GL_n/N]$ is a specialization of the cluster algebra $\mathcal{A}(n)$. Namely, we view $n(n+1)/2$ interval flag minors $\Delta_{\{i, i+1, \dots, i+j-1\}}$ labeling the vertices (i, j) , $i + j \leq n + 1$, $i, j \geq 1$, as formal indeterminates. There are $(n-2)(n-1)/2$ possible mutations out of the seed $Q(n)$, we use the quiver to write the corresponding exchange relations:

$$\Delta_{\{i, \dots, i+j-1\}} \Omega_{ij} = \Delta_{\{i, \dots, i+j-2\}} \Delta_{\{i+1, \dots, i+j\}} \Delta_{\{i-1, \dots, i+j-1\}} + \Delta_{\{i, \dots, i+j\}} \Delta_{\{i+1, \dots, i+j-1\}} \Delta_{\{i-1, \dots, i+j-2\}}, j > 1,$$

and, for $j = 1$, we have

$$\Delta_i \Omega_i = \Delta_{i-1} \Delta_{i, i+1} + \Delta_{i-1, i} \Delta_{i+1, i}, i \geq 2.$$

Using the Plücker relations, we get the new cluster variables $\Omega_i = \Delta_{i-1, i+1}$, $i = 2, \dots, n-1$, and

$$\Omega_{ij} = \Delta_{\{i-1, \dots, i+j-2, i+j\}} \Delta_{\{i+1, \dots, i+j-1\}} - \Delta_{\{i, \dots, i+j-2\}} \Delta_{\{i+1, \dots, i+j\}}.$$

For each new quiver and the corresponding cluster variables we proceed new mutations and so on (we use Plücker relations for calculating cluster variables as we demonstrated above). The resulting set of cluster variables constitute the algebra $\mathcal{P}(n)$ being such a specialization of $\mathcal{A}(n)$. (For $n = 4$ see [10], Section 2.)

Recall that $\mathbb{C}[GL_n/N]$ is the set of regular functions on $GL_n(\mathbb{C})$ which are invariant under the action of N by the right multiplication, that is $\mathbb{C}[GL_n/N] = \mathbb{C}[x_{ij}]^N$.

From the next theorem follows that $\mathcal{P}(n)$ is a cluster algebra on $\mathbb{C}[GL_n/N]$. (Recently, in [13, 14] have shown that the standard quantum deformation of the coordinate ring of a double Bruhat cell is a quantum cluster algebra confirming the Berenstein-Zelevinsky conjecture [3]. Passing to the classical limit, this shows that the coordinate ring of a double Bruhat cell is in fact a genuine cluster algebra.)

Theorem M0.

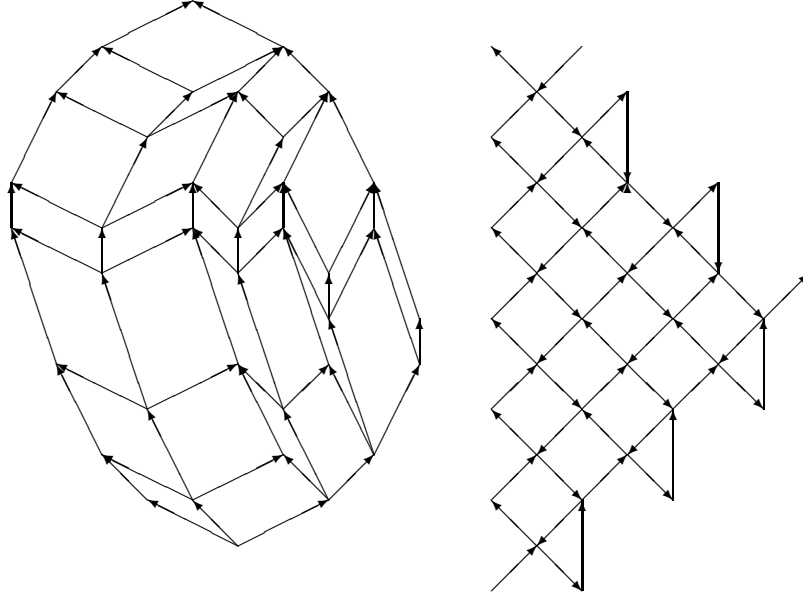
1. Any cluster variable of $\mathcal{P}(n)$ belongs to $\mathbb{C}[x_{ij}]^N$;
2. For every $I \subset [n]$, the flag minor Δ_I is a cluster variable of $\mathcal{P}(n)$.

For the proof of the item 2 see [7]. For the proof of the item 1, we consider a special seed of $\mathcal{P}(n)$. Namely, consider the quiver $T(n)$ corresponding to the pseudo-line arrangement for the following reduced decomposition (see [2] and [7] for algorithms how to associate a quiver to a pseudo-line arrangement)

$$w_0 = \prod_{i=0}^{[(n-1)/2]} s_{i+1}s_{i+2} \dots s_{n-i-1}s_{n-i}s_{n-i-1} \dots s_{i+2}s_{i+1}.$$

An important property of the quiver $T(n)$ is that (i) it can be reached by quiver mutations from $Q(n)$, (ii) the cluster variables (in $\mathcal{P}(n)$) which label the vertices of $T(n)$ are flag minors, and (iii) any seed which is obtained by a single mutation from $T(n)$ has properties (i) and (ii).

For $n = 8$, we draw below a tiling (dual object to pseudo-line arrangement) and the corresponding quiver $T(8)$.



On the left hand side we depicted, for $n = 8$, the tiling corresponding to the above defined reduced decomposition, and on the right hand side is the quiver $T(8)$.

For $T(n)$, the corresponding flag minors are labeled by the following sets $\{1\}, \{1, 2\}, \dots, \{1, 2 \dots n\}$, and pairs $\{2\}, \{2, 3\}, \dots, \{2, 3 \dots, n-1, n\}, \{2, n\}, \{2, 3, n\}, \dots, \{2, 3 \dots, n-2, n\}, \dots, \{k, n-k+3, \dots, n-1, n\}, \{k, k+1, n-k+3, \dots, n-1, n\} \dots, \{k, k+1, \dots, n-k+2, n-k+3, \dots, n-1, n\}, \{k, n-k+2, n-k+3, \dots, n-1, n\}, \{k, k+1, n-k+2, n-k+3, \dots, n-1, n\} \dots, \{k, k+1, \dots, n-k, n-k+2, n-k+3, \dots, n-1, n\}, k = 3, \dots, [n/2]$. One has to assign them from the bottom to top and from the right to the left to the vertices of the tiling and the corresponding quiver $T(n)$.

Let us recall the following facts on $\mathbb{C}[x_{ij}]^N$ (see, for example [11]).

- The coordinate ring $\mathbb{C}[x_{ij}]^N$ is a unique factorization domain.
- For any $I \subset [n]$, the flag minor Δ_I is irreducible polynomial in $\mathbb{C}[x_{ij}]$ (and hence in $\mathbb{C}[x_{ij}]^N$).
- Irreducible flag minors Δ_I are non-equivalent among themselves.

Proof Theorem M0. We prove item 1. We use the seed for the quiver $T(n)$. Suppose there is a cluster variable P which is not a polynomial in flag minors, that is $P \notin \mathbb{C}[x_{ij}]^N$. According to Theorem CL, P is a Laurent polynomial in the variables being the flag minors for the seed with the quiver $T(n)$. Then by assumption, this Laurent polynomial does not belong to $\mathbb{C}[x_{ij}]^N$.

Firstly, we claim that the denominator of this Laurent polynomial can not have multiplies in non-frozen variable. In fact, suppose this not the case and the denominator has a multiplies, say Δ_I , corresponding to a non-frozen vertex v of $T(n)$. Then let us apply the mutation at this vertex. The new seed $T^v(n)$ has cluster variables being also flag minors. The latter set does not contain Δ_I . Since P is a Laurent polynomial in the flag minors of the seed $T^v(n)$, we get that the denominator of this Laurent polynomial does not contain Δ_I . This contradicts to the above facts on $\mathbb{C}[x_{ij}]^N$.

Secondly, we claim that cluster variables can not have denominator with multiplies in the frozen flag minors. We proceed by induction on the number of mutations from the seed $T(n)$. The base of induction is valid by the construction of the seed $T(n)$. Consider the first case when a seed has a cluster variable with the denominator being a monomial in frozen flag minors. This means that for some polynomials S, T, U and $V \in \mathbb{C}[x_{ij}]^N$, we have $\frac{S+T}{U} = \frac{V}{M}$, where M is a monomial in the frozen flag minors. Hence $M(S+T) = VU$. Because of the unique factorization property, we have that U divides M , and U is obtained by the smaller number of mutations than V . Now, we consider the first case in the subsequence of mutations between $T(n)$ and a seed containing U , such that it appears a cluster variable which divides a monomial in frozen flag minor. That is, for some polynomials $S', T', U', V' \in \mathbb{C}[x_{ij}]^N$, it holds that $\frac{S'+T'}{U'} = V'M'$, where M' is a monomial in frozen variables, and S', T', U' do not divide frozen variables. This contradicts that $\mathbb{C}[x_{ij}]^N$ is the unique factorization domain. \square

We need more facts and notions to formulate our main results.

Recall, that a partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_k > 0)$ can be identified with a Young diagram: a left-justified shape of k rows of boxes of lengths $\lambda_1, \dots, \lambda_k$ (numbered from bottom to top for the French style). A semi-standard Young tableau in the alphabet $[n]$ is a filling (assignment a number in $[n]$ to each box) of λ non-decreasing along rows and increasing along columns. Here is an example of a semi-standard Young tableau

$$\begin{array}{c} \boxed{3} \\ \boxed{2} \ \boxed{2} \ \boxed{2} \ \boxed{3} \ \boxed{3} \ \boxed{3} \\ \boxed{1} \ \boxed{1} \ \boxed{1} \ \boxed{1} \ \boxed{2} \ \boxed{2} \ \boxed{2} \ \boxed{3} \ \boxed{3} \ \boxed{3} \ \boxed{3} \end{array}$$

of the shape $(11, 6, 1)$.

There is a bijection between the set of semi-standard Young tableaux and integer \mathbf{D} -tight arrays ([6]). Namely, let us identify the space $\mathbb{R}^{\frac{n(n+1)}{2}}$ and the space of upper triangular $n \times n$ -matrices (a_{ij}) , $1 \leq j \leq i \leq n$. Since we consider the French style to draw Young diagrams, we write matrices in Descartes coordinates. Because of this, such a written matrix

is called an *array*. A typical upper-triangular array A is
$$\begin{pmatrix} 0 & 0 & \dots & 0 & a_{nn} \\ 0 & 0 & \dots & a_{n-1\ n-1} & a_{nn-1} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & a_{22} & \dots & a_{n-1\ 2} & a_{n2} \\ a_{11} & a_{21} & \dots & a_{n-1\ 1} & a_{n1} \end{pmatrix}.$$

For a semi-standard Young tableau T , let us define an array $A(T)$ by the following rule: $a_{ij}(T)$ is equal to the number of boxes in the j -th row filled with i . The array corresponding to the above tableau is $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 3 & 3 \\ 4 & 3 & 4 \end{pmatrix}.$

The set of arrays corresponding the semi-standard Young tableaux is the set of integer points in the cone of **D**-tight arrays. Namely, we have ([6])

Proposition. An (integer) $n \times n$ array A is an array corresponding to a semistandard Young tableau if and only if A belongs to the cone **D**(n) of **D**-tight arrays, that is, for any $j < n$ and $i \in [n]$, it holds

$$a(1, j+1) + \dots + a(i, j+1) \leq a(1, j) + \dots + a(i-1, j).$$

A Gelfand-Tsetlin pattern is a triangular array

$$\begin{array}{ccccccc} x_{nn} & & x_{n-1n} & & x_{n-2n} & \dots & x_{1n} \\ & x_{n-1n-1} & & x_{n-2n-1} & & \dots & x_{1n-1} \\ & & x_{n-2n-2} & & \dots & & x_{1n-2} \\ & & & \ddots & & & \\ & & & & x_{11} & & \end{array}$$

such that $0 \leq x_{ij} \leq x_{i-1\ j-1} \leq x_{i-1\ j}$. The set of all G-T patterns is the cone $GT(n)$.

There is a linear isomorphism sending the cone **D**(n) to the Gelfand-Tsetlin cone $GT(n)$. Namely, this linear isomorphism is: $x_{11} = a_{11}$, $x_{22} = a_{22}$, $x_{12} = a_{11} + a_{21}$, \dots , $x_{ij} = a_{ii} + \dots + a_{ji}$, \dots , $x_{1n} = a_{11} + a_{21} + \dots + a_{n1}$.

It is convenient to us to work with the cone **D**(n). We endow this cone with a tropical semi-ring structure. Namely, the multiplication $\odot = +$ is the sum of arrays, $A \odot B := A + B$. To define the sum, consider the lexicographical order and the corresponding total order on monomials as in Introduction. Then $A \succ B$ iff X^A is bigger than X^B wrt the total order on monomials, where

$$X^A = x_{11}^{a_{11}} x_{21}^{a_{21}} \dots x_{n1}^{a_{n1}} \cdot x_{12}^{a_{12}} x_{22}^{a_{22}} \dots x_{n2}^{a_{n2}} \dots x_{1n}^{a_{1n}} x_{2n}^{a_{2n}} \dots x_{nn}^{a_{nn}}.$$

Then, for a pair of arrays A, B we define $\max(A, B)$ to be equal A if $A \succ B$ and be equal to B , otherwise. Then, we set $A \oplus B = \max(A, B)$.

There is a distinguished basis in $\mathbb{C}[x_{ij}]^N$, the tableau basis. Elements of the *tableau basis* are labeled by semi-standard Young tableaux. Namely, for a semi-standard Young tableau T of shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$, we denote $T_1, \dots, T_{\lambda_1}$ the collections of subsets of $[n]$ being fillings of the columns of T .

For each T_i , we consider the flag minor Δ_{T_i} . Thus, for a semi-standard Young tableau T , we define a monomial in flag minors as follows

$$\Delta_T = \Delta_{T_1} \dots \Delta_{T_{\lambda_1}}.$$

Let us note that the leading term of Δ_T is $X^{A(T)}$.

For the above example of semi-standard Young tableau, the corresponding monomial is $\Delta_{123}(\Delta_{12})^2\Delta_{13}(\Delta_{23})^2\Delta_2(\Delta_3)^4$.

The importance of such monomials is that they form a linear basis in $\mathbb{C}[GL_n/N]$.

Theorem A. ([9, 11]). The set of polynomials Δ_T , while T runs the set of semi-standard Young tableaux filled from the alphabet $[n]$, form a linear basis of $\mathbb{C}[x_{ij}]^N$.

Remark. There are many other linear bases indexed by semi-standard Young tableaux. These bases differ in rules of forming (triangulations of $\mathbf{D}(n)$), for a given tableau T , a monomial in flag minors (see [8]).

Now we formulate main results of the paper.

Theorem M1. Any cluster variable P in $\mathcal{P}(n)$ has the form

$$P = \Delta_T + P',$$

for some semi-standard Young tableau T and some $P' \in \mathbb{C}[x_{ij}]^N$ which is smaller Δ_T wrt the total lexicographical order.

Theorem M2. Let $P = \Delta_T + P'$ and $Q = \Delta_T + Q'$ be cluster variables in $\mathcal{P}(n)$ represented according to Theorem M1 with the same semi-standard Young tableau T . Then

$$P = Q.$$

Because of these theorems, for each seed \mathcal{S} in $\mathcal{P}(n)$, we have a collection of semi-standard Young tableaux, $T_1(\mathcal{S}), \dots, T_{(n-1)(n-2)/2}(\mathcal{S})$, corresponding to mutable variables, and the 'frozen'

$$\text{one-column Young tableaux } T_i^f := \begin{array}{|c|} \hline i \\ \hline \vdots \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}, \quad i = 1, \dots, n \text{ and } T_{n-i}^f := \begin{array}{|c|} \hline n \\ \hline n-1 \\ \hline \vdots \\ \hline i+1 \\ \hline i \\ \hline \end{array}, \quad i = 2, \dots, n.$$

Denote $C(\mathcal{S})$ the cone in $\mathbf{D}(n)$ spanned by the arrays corresponding to this set of semi-standard Young tableaux for the mutable and frozen variables. By the construction the cone $C(\mathcal{S}) \subset \mathbf{D}(n)$ is simplicial. Moreover, the generating set of arrays of this cone is *unimodular*, that is a basis in $\mathbb{Z}^{n(n+1)/2}$. The unimodularity of generators of the cone $C(\mathcal{S})$ follows from: (i) due to the definition $\mathcal{P}(n)$, the vertices of the quiver $Q(n)$ are labeled by the flag minors of one-column Young tableaux filled from intervals; (ii) the the set of arrays corresponding the one-column interval tableaux is unimodular in $\mathbb{Z}^{n(n+1)/2}$; (iii) cluster mutations according to the quiver rule preserve unimodularity (see Remark after the proof of Theorem M1).

Theorem M3. The cones $C(\mathcal{S})$, while \mathcal{S} runs the set of seeds in $\mathcal{P}(n)$, form a simplicial fan in $\mathbf{D}(n)$.

Remark. From Theorem M3 follows that cluster monomials in $\mathbb{C}[GL_n/N]$ are linear independent (for skew-symmetric cluster algebras, linear independence of cluster monomials is proven in [15] using categorification).

For $n = 3, 4, 5$, $\mathbb{C}[GL_n/N]$ is the finite cluster algebra of types A_1 , A_3 and D_6 , respectively. The union of the cones in the fan is the whole $\mathbf{D}(n)$, $n = 3, 4, 5$. Namely, we have

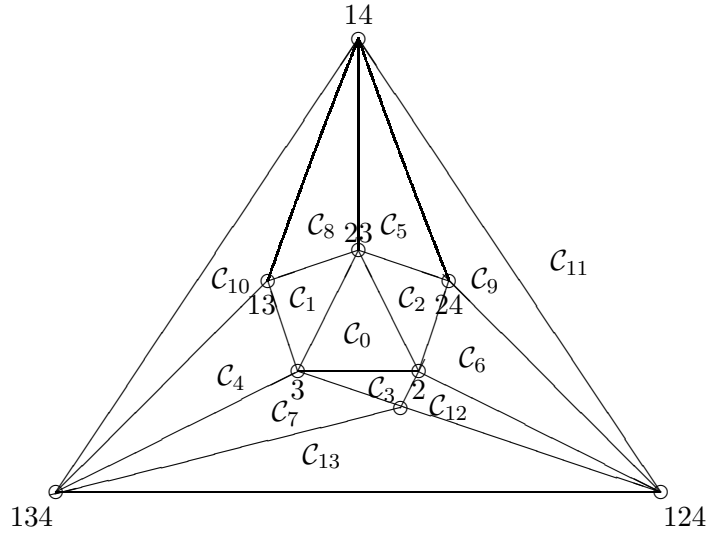
Corollary. For $n = 3, 4, 5$, the union of cones of the fan coincides with $\mathbf{D}(n)$.

For $n \geq 6$ the union of cones of the fan is only a part of $\mathbf{D}(n)$.

We end this section with examples for $n = 4, 5$.

3.1 $n = 4$

Here is the picture on the sphere (see also [10]) in \mathbb{R}^3 (we consider the quotient space by the set of frozen arrays)¹ illustrating the fan of cones of seed of the cluster algebra with the initial seed $Q(4)$ (such an algebra is finite A_3 -algebra).



There are 14 cones (the cone \mathcal{C}_{11} is the triangle 14, 134, 124 on the 2-sphere), 9 variables, 8 of them are of the form A_I , where I is a non-frozen subset of $[4]$ ($2^4 - 1 = 15 - 7 = 8$) and one more variable corresponding to the vertex, indexed by the Young tableau $\begin{array}{|c|} \hline \overline{4} \\ \hline \overline{2} \\ \hline \overline{1} \overline{3} \\ \hline \end{array}$. This Young tableau is a vertex in cones $\mathcal{C}_3, \mathcal{C}_7, \mathcal{C}_{12}, \mathcal{C}_{13}$.

In the quotient space by the frozen arrays, the images of the arrays 2, 3, 24, 14, 23, 13, 134, and 124 are related by the following relations:

$$\begin{aligned} \overline{3} + \overline{14} &= 0 \text{ (because there holds } 3+14=1+34 \text{ in } \mathbf{D}(n)); \overline{2} + \overline{134} = 0 \text{ (} 2+134=1+234); \\ \overline{23} + \overline{124} &= 0 \text{ (} 23+124=12+234); \overline{2} + \overline{14} = \overline{24} \text{ (} 2+14=1+24); \overline{23} + \overline{134} = \overline{13} \text{ (} 23+134=13+234). \end{aligned}$$

$$\text{For unique non-extreme ray cluster variable we have } \begin{array}{c} \overline{4} \\ \hline \overline{2} \quad \overline{3} \\ \hline \end{array} = \overline{124} + \overline{3}.$$

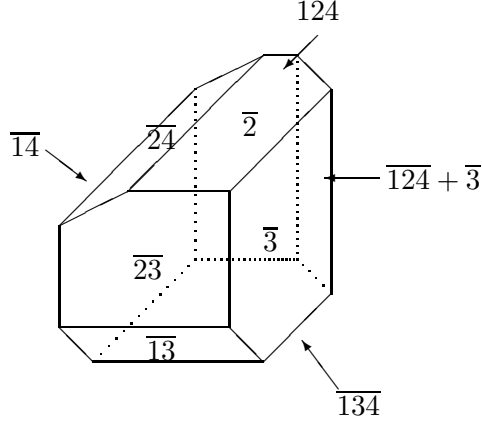
Let the vectors $\overline{3}$, $\overline{23}$ and $\overline{2}$ be a basis of the 3-dim quotient space. Then these cluster vectors are of the form

$$\overline{3} = (1, 0, 0), \overline{14} = (-1, 0, 0), \overline{23} = (0, 1, 0), \overline{124} = (0, -1, 0), \overline{2} = (0, 0, 1), \overline{134} = (0, 0, -1),$$

¹ Let $\phi : \mathbb{R}^{n(n+1)/2} \rightarrow \mathbb{R}^{(n-2)(n-1)/2}$ be the quotient by frozen arrays $A(T_i^f)$ and $A(T_{n-i}^f)$. Then it is easy to verify that the collection $\phi(C(\mathcal{S}))$, \mathcal{S} runs the cluster seeds, is a simplicial fan.

$$\overline{24} = (-1, 0, 1), \overline{13} = (0, 1, -1), \text{ and } \overline{\begin{smallmatrix} 4 \\ 2 \\ 1 \quad 3 \end{smallmatrix}} = (1, -1, 0).$$

This fan is the normal fan to the associahedron



3.2 n=5

For $n = 5$ the fan has 672 simplicial cones and 36 one-dimensional rays corresponding to mutable variables and 9 rays corresponding to frozen. The quotient of this fan by frozen rays is the normal fan to the D_6 -associahedron. This is 6-dimensional polytope which has 672 vertices and 36 facets.

The mutable variables have the leading term Δ_T (Theorem M1) of the following form: either T is an one column semistandard Young tableaux whose filling is a subset $I \subset [5]$, such that I is not an interval containing either 1 or n , (in such a case, cluster variables coincide with the leading terms) or T is one of the following set of two column tableaux:

$$\begin{array}{cccccccccccc} 4 & & 5 & & 5 & & 5 & & 5 & & 4 & & 5 & & 5 & & 5 & & 5 \\ 2 & & 2 & & 2 & & 3 & & 3 & & 2 \ 5 & & 3 \ 4 & & 2 & & 2 \ 4 & & 2 \ 4 \\ 1 \ 3 & & 1 \ 3 & & 1 \ 4 & & 2 \ 4 & & 1 \ 4 & & 1 \ 3 & & 1 \ 2 & & 1 \ 4 & & 1 \ 3 & & 1 \ 2 \end{array},$$

$$\begin{array}{ccc} 5 & 5 & 5 \\ 4 & 4 & 4 \\ 2 & 2 \ 4 & 2 \ 5 \\ 1 \ 3 & 1 \ 3 & 1 \ 3 \end{array}.$$

4 Proof of Theorems M1-M3

Proof of Theorem M1. Because of Theorem M0 (item 1), each cluster variable is a polynomial in flag minors. So, we have to prove that each such cluster polynomial P has the coefficient 1 with the leading monomial wrt the lexicographic order. Proceed by induction on the number of mutations from the initial seed with the quiver $Q(n)$. For one mutations we gave explicit formulae for Ω_{ij} and Ω_i above, and we are done.

Suppose we are done for some number of mutations, and consider one more mutation.

Consider the quiver mutation rule

$$x_v^{new} \cdot x_v = \prod_{v' \in In(v)} x_{v'}^{w(v'v)} + \prod_{v'' \in Out(v)} x_{v''}^{w(v,v'')}.$$

By induction, for each $v, v' \in In(v)$, and $v'' \in Out(v)$, we have $x_v = \Delta_{T(v)} + P'_v$, $x_{v'} = \Delta_{T(v')} + P'_{v'}$ and $x_{v''} = \Delta_{T(v'')} + P'_{v''}$. By Theorem M0, $x_v^{new} = a\Delta_T + P'$. Then, because the product of leading terms on the left hand side equals the product of the leading terms on the right hand side, we have

$$aX^{A(T)}X^{A(T(v))} = X^{\max(\sum_{v' \in In(v)} w(v',v)A(T(v')), \sum_{v'' \in Out(v)} w(v,v'')A(T(v'')))},$$

where the operation \max for the cone $\mathbf{D}(n)$ is defined above.

Because of this, we have $a = 1$. □

Remark. Let us note that from the proof of this theorem, for any seed of $\mathcal{P}(n)$ and any vertex v , we have

$$\max(\sum_{v' \in In(v)} w(v',v)A(T(v')), \sum_{v'' \in Out(v)} w(v,v'')A(T(v''))) - A(T(v)) \in \mathbf{D}(n). \quad (4.1)$$

From this relation follows that if, for a seed \mathcal{S} , the set of arrays $\{A(T_v)\}$ (vectors in $\mathbb{Z}^{n(n+1)/2}$), v runs the vertices of the quiver for \mathcal{S} , is unimodular set of vectors in $\mathbb{Z}^{n(n+1)/2}$. Then, for any seed, which is obtained by a mutation, and hence, for all seeds in $\mathcal{P}(n)$, the unimodularity is valid. In fact, due to (4.1), the transformation matrix for mutations of cluster cones is unimodular. For the initial quiver $Q(n)$, the interval tableaux, the labels of the vertices of $Q(n)$, form a unimodular set. Therefore every cone $C(\mathcal{S})$ is unimodular.

Proof of Theorem M2. Suppose there are two seeds \mathcal{S} and \mathcal{S}' and cluster variables w and w' in these seeds such that $w = \Delta_T + P$ and $w' = \Delta_T + P'$, where P and P' are polynomials in the flag minors which are smaller (wrt the lexicographical order) Δ_T .

Suppose $w \neq w'$. By Theorem CL, w is a Laurent polynomial of the cluster variables of the seed \mathcal{S}' . We claim that this Laurent polynomial has the form

$$w = w' + p'(S' \setminus w'), \quad (4.2)$$

where $p'(S' \setminus w')$ is a Laurent polynomial in variables of the seed \mathcal{S}' except w' . In fact, the leading term of w is $X^{A(T)}$. Then there exists a Laurent monomial in variables S' of the above Laurent polynomial, such that its leading term (that is the product of leading term of this monomial) is equal to $X^{A(T)}$. But the leading term of w' is also $X^{A(T)}$. Hence there is a linear dependence between arrays for tableaux corresponding to the variables of \mathcal{S}' . Due to Remark after proof Theorem M1, this is not the case. Thus (4.2) holds true.

By the same line of arguments, it holds that

$$w' = w + p(S \setminus w),$$

where $p(S \setminus w)$ is a Laurent polynomial in variables of the seed $\mathcal{S} \setminus w$.

Therefore, we have

$$p(S \setminus w) + p'(S' \setminus w') = 0.$$

Because of the positivity of coefficients of Laurent polynomials (Theorem CP), the latter equality is possible if and only if

$$p = p' = 0.$$

In fact, the numerator of $p(S \setminus w) + p'(S' \setminus w')$ is a Laurent polynomial in variables of the initial seed (with the quiver $Q(n)$) with positive coefficients. Since, for a totally positive matrix, all interval minors are positive we get $p(S \setminus w) + p'(S' \setminus w') > 0$. Thus, $p = p' = 0$, and hence $w = w'$. \square

Proof of Theorem M3.

Suppose there are two seeds, \mathcal{S} and \mathcal{S}' , such that the cones $C := C(\mathcal{S})$ and $C' := C(\mathcal{S}')$ intersect such that it holds $\text{int}(C) \cap \text{int}(C') \neq \emptyset$ ($\text{int}(C')$ denotes interior of a cone). Then, because the cones are unimodular, this intersection contains an integer array, say, corresponding a semi-standard Young tableau T . Then there is a cluster monomial y in the variables of the seed \mathcal{S} such that $y = x^{A(T)} + \text{lexicographically smaller terms}$, and there is a cluster monomial y' in the variables of the seed \mathcal{S}' such that $y' = x^{A(T)} + \text{lexicographically smaller terms}$.

Let us show that $y = y'$. By Theorem CL, the cluster monomial y is the product of Laurent polynomials in the variables of the seed \mathcal{S}' . The same reasoning as in the proof Theorem M2 shows that $y = y' + R(\mathcal{S}')$. Similarly, we get $y' = y + R'(\mathcal{S})$, where R and R' are Laurent polynomials with positive coefficients (Theorem CP). Thus, $R(C') + R(C) = 0$ is possible iff $R(C') = R(C) = 0$. Hence $y = y'$.

Now, recall that $y = \prod_v s_v^{a_v}$ is the monomial in the cluster variables of \mathcal{S} . By Theorem CL, y is the product of the Laurent polynomials in cluster variables of \mathcal{S}' , since each s_v is a Laurent polynomial in \mathcal{S}' . But $y = y'$ and y' is a monomial in \mathcal{S}' . Hence each cluster variable s_v is a Laurent monomial in \mathcal{S}' . And, similarly, each cluster variable in \mathcal{S}' is a Laurent monomial in \mathcal{S} .

Since the cones $C := C(\mathcal{S})$ and $C' := C(\mathcal{S}')$ have a common interior point, there is a facet of one of this cones which has a common interior point with another cone. Let this facet be a facet of $C := C(\mathcal{S})$. Note that the vertices corresponding to the frozen arrays belong to this facet. Let \mathcal{S}'' be a seed obtained through the mutation of \mathcal{S} in the vertex, corresponding to the ray of $C := C(\mathcal{S})$ which does not belong to this facet. Then $C'' := C(\mathcal{S}'')$ and $C' := C(\mathcal{S}')$ have a common interior point (for example, in neighbor of the facet). Hence, by the same reasoning as above, the new cluster variable in \mathcal{S}'' is a Laurent monomial in cluster variable in \mathcal{S}' , but this is not the case, because all cluster variables in \mathcal{S} are Laurent monomials in \mathcal{S}' . \square

5 Lusztig basis

Let $X \in \text{Mat}_q(n)$ be a $n \times n$ quantum q^2 -matrix. That is, the following relations hold:

$$x_{il}x_{ik} = q^2x_{ik}x_{il} \quad \forall i, \forall k < l; \quad x_{jk}x_{ik} = q^2x_{ik}x_{jk} \quad \forall i < j, \forall k; \quad x_{jk}x_{il} = x_{il}x_{jk} \quad \forall i < j, \forall k < l; \quad x_{jl}x_{ik} = x_{ik}x_{jl} + (q^2 - q^{-2})x_{il}x_{jk} \quad \forall i < j, \forall k < l.$$

Consider monomials X^A , $A \in \mathbf{D}(n)$, recall that the product is taken w.r.t. the lexicographical order \succ . Define $l(A) := \sum_i (\sum_{j>k} a_{ij}a_{ik} + \sum_{j>k} a_{ji}a_{ki})$ and set $X(A) := q^{-l(A)}X^A$.

Lusztig in ([17]) proved that there exists a unique basis in $\mathbb{C}_q[GL_n/N]$ (the dual canonical basis) $\mathbf{B}^*(n) = \{b_A, A \in \mathbf{D}(n)\}$ determined by the following conditions:

- $\overline{b_A(X)} = b_A(X)$, where $\overline{qX_{ij}X_{lk}} = q^{-1}X_{lk}X_{ij}$;
- $b_A(X) = X(A) + \sum_{B \prec A} c_A(B)X(B)$, where $B \prec A$ denotes the above considered lexicographical order on the set of the arrays, and the summation is taken over the matrices B with the same column and row sums as those in A , and $c_A(B) \in q\mathbb{Z}[q]$.

Because of this definition, for $q = 1$, $b_A(X) = \Delta_{T(A)} + b'(X)$, where $T(A)$ is the semi-standard Young tableau corresponding to A , and $\Delta_{T(A)} \succ b'$. Thus, the Lusztig basis is another basis of $\mathbb{C}[GL_n/N]$ labeled by the semi-standard Young tableaux the (integer points of the cone $\mathbf{D}(n)$).

Let us say that an element $b \in \mathbf{B}^*(n)$ is *real* if, for any r , $b^r \in \mathbf{B}^*(n)$ up to a power of q . In such a case, $b_A^r = q^{r l(A)} b_{rA}$.

We call a semi-standard Young tableau *real* if it labels the real element of $\mathbf{B}^*(n)$.

Conjecture. The labels of the cluster monomials in $\mathbb{C}[GL_n/N]$ constitute the set of real semi-standard Young tableaux.

References

- [1] A. Berenstein, S. Fomin, and A. Zelevinsky, Parametrization of canonical bases and totally positive matrices, *Adv. Math.* **122** (1996) 49–149.
- [2] A. Berenstein, S. Fomin, and A. Zelevinsky, Cluster algebras III: Upper bounds and double Bruhat cells, arXiv:math/0305434v3
- [3] A. Berenstein and A. Zelevinsky, Quantum cluster algebras, *Adv. Math.* **195** (2005) 405–455.
- [4] S. Fomin and A. Zelevinsky, Cluster Algebras. I. Foundations, *J. Amer. Math. Soc.* **15** (2002) 497–529.
- [5] Fomin, Sergey; Zelevinsky, Andrei Cluster algebras. II. Finite type classification. *Invent. Math.* 154 (2003), no. 1, 63–121
- [6] V.I. Danilov and G.A. Koshevoy, Arrays and Combinatorics of Young Tableaux, *Uspehi Matematicheskikh Nauk*, 60:2(2005), 79–142, (in Russian), (translation in *Russian Math. Surveys* 60 (2005), no. 2, 269–334)
- [7] V. Danilov, A. Karzanov, and G. Koshevoy, Generalized tilings and Plücker cluster algebras, *Lattice Path Combinatorics*, University Siena, 2010
- [8] V. Danilov, A. Karzanov, and G. Koshevoy, Coherent fans in the space of flows in framed graphs, DIMACS (Proceedings of FPSAC’12).
- [9] J. Désarménien, J. P. S. Kung, G.-C. Rota, Invariant theory, Young bitableaux and Combinatorics, *Adv. Math.* **27** (1978), 63–92
- [10] S.Fomin, Total positivity and cluster algebras, Proceedings of the International Congress of Mathematicians, Hyderabad, India, 2010, arXiv:1005.1086[math.RA]
- [11] W.Fulton, Young tableau, representation theory and geometry, Cambridge University Press, 1997
- [12] I. M. Gelfand and A. Zelevinsky, Canonical basis in irreducible representations of gl_3 and its applications, in: Group theoretical methods in physics, Vol. II (Jurmala, 1985), 127146, VNU Sci. Press, Utrecht, 1986.

- [13] K. Goodreal and M.Yakimov, Quantum cluster algebra structures on quantum nilpotent algebras, arXiv:1309.7869
- [14] K. Goodreal and M.Yakimov, The Berenstein-Zelevinsky cluster algebra conjecture, *in preparation*
- [15] Giovanni Cerulli Irelli, Bernhard Keller, Daniel Labardini-Fragoso, Pierre-Guy Plamondon, Linear independence of cluster monomials for skew-symmetric cluster algebras, arXiv:1203.1307
- [16] K. Lee and R. Schiffler, Positivity for cluster algebras, ArXiv:1306.2415v2
- [17] G. Lusztig, Canonical bases arising from quantized enveloping algebras, J. Amer. Math. Soc. 3 (1990) 447498.
- [18] Miller, Ezra; Sturmfels, Bernd Combinatorial commutative algebra. Graduate Texts in Mathematics, 227. Springer-Verlag, New York, 2005
- [19] J. Scott, Grassmannians and cluster algebras, *Proc. Lond. Math. Soc.* (3) 92 (2) (2006) 345380.